Periodic solutions and exponential stability in delayed cellular neural networks

Jinde Cao*
Adult Education College, Yunnan University, Kunming 650091, People's Republic of China

(Received 7 April 1999)

Some simple sufficient conditions are given ensuring global exponential stability and the existence of periodic solutions of delayed cellular neural networks (DCNNs) by constructing suitable Lyapunov functionals and some analysis techniques. These conditions are easy to check in terms of system parameters and have important leading significance in the design and applications of globally stable DCNNs and periodic oscillatory DCNNs. In addition, two examples are given to illustrate the theory. [S1063-651X(99)05909-7]

PACS number(s): 87.10.+e, 85.40.Ls, 05.45.−a, 43.80.+p

I. INTRODUCTION

It is well known that dynamic behavior of neural networks play an important role in the design and applications of neural networks. Cellular neural networks (CNNs) are formed by many units called cells. The structure of the CNN is similar to that found in cellular automata, namely, any cell in a cellular neural network is connected only to its neighbor cells. A cell contains linear and nonlinear circuit elements, which typically are linear capacitors, linear resistors, linear and nonlinear controlled sources, and independent sources. The circuit diagram and connection pattern implementing for the CNN can be found in Refs. [1,2]. Processing of moving images requires the introduction of delay in the signals transmitted among the cells [3]. Some results of stability for CNNs and DCNNs can be found in Refs. [1,2,13] and Refs. [3–9,11,12,14,16], respectively, and the references cited therein. In this paper, we investigated further a class of CNN with delays (DCNN), which can be described by delayed differential equations (namely, functional differential equations). To the best of my knowledge, few authors have considered global exponential stability and periodic solutions for the DCNN. The purpose of this paper is to give some simple sufficient conditions for global exponential stability and the existence of periodic solutions of the DCNN by constructing suitable Lyapunov functionals and some analysis techniques. These possess important leading significance in the design and applications of globally stable DCNNs and periodic oscillatory DCNNs, and are of great interest in many applications. In addition, two examples are given to illustrate the theory.

In this paper, we study the global exponential stability and periodic solutions of the DCNN model described by differential equations with delays

$$x_i'(t)=-c_ix_i(t)+\sum_{j=1}^{n} a_{ij}f_j(x_j(t))+\sum_{j=1}^{n} b_{ij}f_j(x_j(t-\tau_j))+I_i(t),c_i>0,\quad i=1,2,\dots,n \tag{1}$$

in which n corresponds to the number of units in a neural network, $x_i(t)$ corresponds to the state of the ith unit at time t, $f_j[x_j(t)]$ denotes the output of the jth unit at time t, a_{ij},b_{ij},c_i are constant, a_{ij} denotes the strength of the jth unit on the ith unit at time t, b_{ij} denotes the strength of the jth unit on the ith unit at time $t-\tau_j$, $I_i(t)$ denotes the external bias on the ith unit at time t, τ_j corresponds to the transmission delay along the axon of the jth unit and is not a negative constant, and c_i represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs.

In the following, we assume that each of the relations between the output of the cell f_i $(i=1,2,\dots,n)$ and the state of the cell possess following properties: H1. $f_i(i=1,2,\dots,n)$ is bounded on R and H2. There is a number $\mu_i>0$ such that $|f_i(u)-f_i(v)|\leqslant\mu_i|u-v|$ for any $u,v\in\mathbb{R}$.

It is easy to find from hypothesis H2 that f_i is a continuous function on R. In particular, if the relation between the output of the cell and the state of the cell is described by a piecewise-linear function $f_i(x)=\frac{1}{2}(|x+1|-|x-1|)$, then it is easy to see that the function f_i clearly satisfies the hypotheses H1 and H2 above, and $\mu_i\equiv 1$ $(i=1,2,\dots,n)$.

II. GLOBAL EXPONENTIAL STABILITY OF THE DCNN

Consider the special case of the DCNN model (1) as $I_i(t)=I_i$, i.e.,

$$x_i'(t)=-c_ix_i(t)+\sum_{j=1}^{n} a_{ij}f_j[x_j(t)]+\sum_{j=1}^{n} b_{ij}f_j[x_j(t-\tau_j)]+I_i,c_i>0,\quad i=1,2,\dots,n, \tag{2}$$

where the delays $\tau_i,i=1,2,\dots,n$, are non-negative constants, $I_i,i=1,2,\dots,n$ are constant numbers.

Assume that the nonlinear system (2) is supplemented with initial values of the type

$$x_i(t)=\phi_i(t),-\tau\leqslant t\leqslant 0,\tau=\max_{1\leqslant i\leqslant n}\tau_i,\quad i=1,2,\dots,n$$

in which $\phi_i(t),i=1,2,\dots,n$, are continuous functions, and the system (2) has a unique equilibrium $x^*=(x_1^*,x_2^*,\dots,x_n^*)$. Let $x^*=(x_1^*,x_2^*,\dots,x_n^*)$ be the equilibrium of system (2), we denote

*Electronic address: jdcao@ynu.edu.cn

$$\|\phi-x^*\|=\sup_{-\tau\le t\le 0}\left[\sum_{i=1}^{n}|\phi_i(t)-x_i^*|\right].$$

Definition 1. The equilibrium $x^*=(x_1^*,x_2^*,\cdots,x_n^*)$ is said to be globally exponentially stable, if there exist constants $\varepsilon>0$ and $M\ge 1$ such that

$$\sum_{i=1}^{n}|x_i(t)-x_i^*|\le M\|\phi-x^*\|e^{-\varepsilon t}$$

for all $t\ge 0$.

Theorem 1. For the DCNN (2), suppose that the outputs of the cell $f_i(i=1,2,\ldots,n)$ satisfy the hypotheses H1 and H2 above and there exist constants $\alpha_j>0, j=1,2,\ldots,n$, such that

$$-c_j+\frac{\mu_j}{\alpha_j}\sum_{i=1}^{n}\alpha_i|a_{ij}|+\frac{\mu_j}{\alpha_j}\sum_{i=1}^{n}\alpha_i|b_{ij}|<0,\quad j=1,2,\ldots,n$$

in which $\mu_j(j=1,2,\ldots,n)$ is constant numbers of the hypotheses H2 above. Then the equilibrium x^* is globally exponentially stable.

Proof. Since

$$-c_j+\frac{\mu_j}{\alpha_j}\sum_{i=1}^{n}\alpha_i|a_{ij}|+\frac{\mu_j}{\alpha_j}\sum_{i=1}^{n}\alpha_i|b_{ij}|<0,\quad j=1,2,\ldots,n.$$

We can choose a small $\varepsilon>0$ such that

$$\varepsilon-c_j+\frac{\mu_j}{\alpha_j}\sum_{i=1}^{n}\alpha_i|a_{ij}|+\frac{e^{\varepsilon\tau}\mu_j}{\alpha_j}\sum_{i=1}^{n}\alpha_i|b_{ij}|<0,$$
$$j=1,2,\ldots,n.$$

We rewrite Eq. (2) as

$$[x_i(t)-x_i^*]'=-c_i[x_i(t)-x_i^*]+\sum_{j=1}^{n}a_{ij}[f_j(x_j(t))-f_j(x_j^*)]$$
$$+\sum_{j=1}^{n}b_{ij}[f_j(x_j(t-\tau_j))-f_j(x_j^*)]. \tag{3}$$

Now consider the Lyapunov functional

$$V(t)=\sum_{i=1}^{n}\alpha_i\left[|x_i(t)-x_i^*|e^{\varepsilon t}+\sum_{j=1}^{n}|b_{ij}|\int_{t-\tau_j}^{t}|f_j(x_j(s))-f_j(x_j^*)|e^{\varepsilon(s+\tau_j)}ds\right].$$

Calculating the upper right derivate D^+V of V along the solution of Eq. (3), we have

$$\begin{aligned}D^+V(t)|_{(3)}&=\sum_{i=1}^{n}\alpha_i\left[D^+[|x_i(t)-x_i^*|e^{\varepsilon t}]|_{(3)}+\sum_{j=1}^{n}|b_{ij}||f_j[x_j(t)]-f_j(x_j^*)|e^{\varepsilon(t+\tau_j)}-\sum_{j=1}^{n}|b_{ij}||f_j[x_j(t-\tau_j)]-f_j(x_j^*)|e^{\varepsilon t}\right]\\
&\le\sum_{i=1}^{n}\alpha_i\left[(\varepsilon-c_i)|x_i(t)-x_i^*|e^{\varepsilon t}+e^{\varepsilon t}\sum_{j=1}^{n}|a_{ij}||f_j[x_j(t)]-f_j(x_j^*)|+e^{\varepsilon t}e^{\varepsilon\tau}\sum_{j=1}^{n}|b_{ij}||f_j[x_j(t)]-f_j(x_j^*)|\right]\\
&\le e^{\varepsilon t}\sum_{i=1}^{n}\alpha_i\left[(\varepsilon-c_i)|x_i(t)-x_i^*|+\sum_{j=1}^{n}|a_{ij}||f_j[x_j(t)]-f_j(x_j^*)|+e^{\varepsilon\tau}\sum_{j=1}^{n}|b_{ij}||f_j[x_j(t)]-f_j(x_j^*)|\right]\\
&\le e^{\varepsilon t}\sum_{i=1}^{n}\alpha_i\left[(\varepsilon-c_i)|x_i(t)-x_i^*|+\sum_{j=1}^{n}|a_{ij}|\mu_j|x_j(t)-x_j^*|+e^{\varepsilon\tau}\sum_{j=1}^{n}|b_{ij}|\mu_j|x_j(t)-x_j^*|\right]\\
&\le e^{\varepsilon t}\sum_{j=1}^{n}\alpha_j\left[\varepsilon-c_j+\frac{\mu_j}{\alpha_j}\sum_{i=1}^{n}\alpha_i|a_{ij}|+\frac{e^{\varepsilon\tau}\mu_j}{\alpha_j}\sum_{i=1}^{n}\alpha_i|b_{ij}|\right]|x_j(t)-x_j^*|\\
&\le 0\end{aligned}$$

and so

$$V(t)\le V(0),\quad t\ge 0$$

since

$$e^{\varepsilon t}(\min_{1\le j\le n}\alpha_j)\sum_{i=1}^{n}|x_i(t)-x_i^*|\le V(t),\quad t\ge 0$$

$$\begin{aligned}V(0)&=\sum_{i=1}^{n}\alpha_i\left[|\phi_i(0)-x_i^*|+\sum_{j=1}^{n}|b_{ij}|\int_{-\tau_j}^{0}|f_j[x_j(s)]-f_j(x_j^*)|e^{\varepsilon(s+\tau_j)}ds\right]\\
&\le\left[\max_{1\le i\le n}\alpha_i+\mu\tau e^{\varepsilon\tau}\sum_{i=1}^{n}\alpha_i\max_{1\le j\le n}(|b_{ij}|)\right]\\
&\quad\times\|\phi-x^*\|,\end{aligned}$$

where $\mu=\max_{1\leqslant j\leqslant n}(\mu_j)$ are constants. Then we easily get

$$\sum_{i=1}^{n}|x_i(t)-x_i^*|\leqslant M||\phi-x^*||e^{-\varepsilon t}$$

for all $t\geqslant 0$, where $M\geqslant 1$ is a constant. This implies that the equilibrium $x^*=(x_1^*,x_2^*,\ldots,x_n^*)$ is globally exponentially stable. Applying Theorem 1 above, we can easily prove the following theorems 2 and 3.

Theorem 2. For the DCNN (2), suppose that the outputs of the cell $f_i(i=1,2,\ldots,n)$ satisfy the hypotheses H1 and H2 above and

$$c_j>\sum_{i=1}^{n}\mu_j|a_{ij}|+\sum_{i=1}^{n}\mu_j|b_{ij}|,\quad j=1,2,\ldots,n$$

in which μ_j $(j=1,2,\ldots,n)$ is constant numbers of the hypotheses H2 above. Then the equilibrium x^* is also globally exponentially stable.

Theorem 3. If the relation between the output of the cell and the state of the cell is described by a piecewise-linear function $f_i(x)=\frac{1}{2}(|x+1|-|x-1|)$ and

$$c_j>\sum_{i=1}^{n}|a_{ij}|+\sum_{i=1}^{n}|b_{ij}|,\quad j=1,2,\ldots,n,$$

then the equilibrium x^* is also globally exponentially stable.

III. PERIODIC SOLUTIONS OF DCNN

In this section, we study the periodic solutions of the DCNN of the type

$$x_i'(t)=-c_ix_i(t)+\sum_{j=1}^{n}a_{ij}f_j[x_j(t)]+\sum_{j=1}^{n}b_{ij}f_j(x_j(t-\tau_j))+I_i(t),c_i>0,\quad i=1,2,\ldots,n \tag{4}$$

in which $I_i:R^+\rightarrow R,i=1,2,\ldots,n$, are continuously periodic functions with period ω, i.e., $I_i(t+\omega)=I_i(t)$. Other symbols possess the same meaning as that of Eq. (2).

Theorem 4. For the DCNN (4), suppose that the outputs of the cell $f_i(i=1,2,\ldots,n)$ satisfy the hypotheses H1 and H2 above and there exist constants $\alpha_j>0,j=1,2,\ldots,n$, such that

$$-c_j+\frac{\mu_j}{\alpha_j}\sum_{i=1}^{n}\alpha_i|a_{ij}|+\frac{\mu_j}{\alpha_j}\sum_{i=1}^{n}\alpha_i|b_{ij}|<0,\quad j=1,2,\ldots,n$$

in which $\mu_j(j=1,2,\ldots,n)$ is constant numbers of the hypotheses H2 above. Then there exists exactly one ω-periodic solution of Eq. (4) and all other solutions of Eq. (4) converge exponentially to it as $t\rightarrow+\infty$.

Proof. Let $C=C([-\tau,0],\mathbb{R}^n)$ be the Banach space of continuous functions which map $[-\tau,0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. For any $\varphi\in C$, we define

$$||\varphi||=\sup_{-\tau\leqslant\theta\leqslant 0}|\varphi(\theta)|,$$

in which $|\varphi(\theta)|=\Sigma_{i=1}^{n}|\varphi_i(\theta)|$.

For $\forall\phi,\psi\in C$, we denote the solutions of Eq. (4) through $(0,\phi)$ and $(0,\psi)$ as

$$x(t,\phi)=(x_1(t,\phi),x_2(t,\phi),\ldots,x_n(t,\phi))^T,x(t,\psi)=(x_1(t,\psi),x_2(t,\psi),\ldots,x_n(t,\psi))^T,$$

respectively.

Define

$$x_t(\phi)=x(t+\theta,\phi),\theta\in[-\tau,0],\quad t\geqslant 0,$$

then $x_t(\phi)\in C$ for $\forall$ $t\geqslant 0$.

Thus we follow from system (4) that

$$\begin{aligned}(x_i(t,\phi)-x_i(t,\psi))'&=-c_i(x_i(t,\phi)-x_i(t,\psi))+\sum_{j=1}^{n}a_{ij}[f_j(x_j(t,\phi))\\&-f_j(x_j(t,\psi))]+\sum_{j=1}^{n}b_{ij}[f_j(x_j(t-\tau_j,\phi))\\&-f_j(x_j(t-\tau_j,\psi))]\end{aligned}$$

for $t\geqslant 0,i=1,2,\ldots,n$. We choose a small $\varepsilon>0$ such that

$$\varepsilon-c_j+\frac{\mu_j}{\alpha_j}\sum_{i=1}^{n}\alpha_i|a_{ij}|+\frac{e^{\varepsilon\tau}\mu_j}{\alpha_j}\sum_{i=1}^{n}\alpha_i|b_{ij}|<0,\quad j=1,2,\ldots,n.$$

We consider the Lyapunov functional

$$V(t)=\sum_{i=1}^{n}\alpha_i\left[|x_i(t,\phi)-x_i(t,\psi)|e^{\varepsilon t}+\sum_{j=1}^{n}|b_{ij}|\int_{t-\tau_j}^{t}|f_j(x_j(s,\phi))-f_j(x_j(s,\psi))|e^{\varepsilon(s+\tau_j)}ds\right].$$

By a minor modification of the proof of theorem 1, we can easily get

$$\sum_{i=1}^{n}|x_i(t,\phi)-x_i(t,\psi)|\leqslant ke^{-\varepsilon t}||\phi-\psi||$$

for $\forall t\geqslant 0$, where $k\geqslant 1$ is a constant. One can easily follow from the formula above that

$$||x_t(\phi)-x_t(\psi)||\leqslant ke^{-\varepsilon(t-\tau)}||\phi-\psi||. \tag{5}$$

We can choose a positive integer m such that

$$ke^{-\varepsilon(m\omega-\tau)}\leqslant\frac{1}{4}.$$

Now define a Poincare mapping P: $C\rightarrow C$ by $P\phi=x_\omega(\phi)$. Then we can derive from Eq. (4) that

$$||P^m\phi - P^m\psi|| \leq \tfrac{1}{4}||\phi-\psi||.$$

This implies that P^m is a contraction mapping, hence there exists a unique fixed point $\phi^*\in C$ such that $P^m\phi^*=\phi^*$. Note that

$$P^m(P\phi^*)=P(P^m\phi^*)=P\phi^*.$$

This shows that $P\phi^*\in C$ is also a fixed point of P^m, so $P\phi^*=\phi^*$, i.e.,

$$x_\omega(\phi^*)=\phi^*.$$

Let $x(t,\phi^*)$ be the solution of Eq. (4) through $(0,\phi^*)$. Obviously, $x(t+\omega,\phi^*)$ is also a solution of Eq. (4), and note that

$$x_{t+\omega}(\phi^*)=x_t(x_\omega(\phi^*))=x_t(\phi^*)$$

for $t\geq 0$, therefore

$$x(t+\omega,\phi^*)=x(t,\phi^*)$$

for $t\geq 0$.

This shows that $x(t,\phi^*)$ is exactly one ω-periodic solution of Eq. (4), and it easy to see that all other solutions of Eq. (4) converge exponentially to it as $t\to+\infty$. Applying theorem 4 above, we can prove the following theorems.

Theorem 5. For the DCNN (4), suppose that the outputs of the cell $f_i(i=1,2,\ldots,n)$ satisfy the hypotheses H1 and H2 above and

$$c_j>\sum_{i=1}^{n}\mu_j|a_{ij}|+\sum_{i=1}^{n}\mu_j|b_{ij}|,\quad j=1,2,\ldots,n$$

in which $\mu_j(j=1,2,\ldots,n)$ is constant numbers of the hypotheses H2 above. Then there exists exactly one ω-periodic solution of Eq. (4) and all other solutions of Eq. (4) converge exponentially to it as $t\to+\infty$.

Theorem 6. If the relation between the output of the cell and the state of the cell is described by a piecewise-linear function $f_i(x)=\frac{1}{2}(|x+1|-|x-1|)$ and

$$c_j>\sum_{i=1}^{n}|a_{ij}|+\sum_{i=1}^{n}|b_{ij}|,\quad j=1,2,\ldots,n.$$

Then there exists exactly one ω-periodic solution of Eq. (4) and all other solutions of Eq. (4) converge exponentially to it as $t\to+\infty$.

IV. EXAMPLES

Example 1. Consider the cellular neural networks with delays

$$\begin{cases} x_1'(t)=-9x_1(t)+2f(x_1(t))-f(x_2(t))+2f(x_1(t-\tau_1))+f(x_2(t-\tau_2))+I_1, \\ x_2'(t)=-9x_2(t)-f(x_1(t))+2f(x_2(t))+f(x_1(t-\tau_1))+2f(x_2(t-\tau_2))+I_2, \end{cases} \tag{6}$$

where the relation between the output of the cell and the state of the cell is described by a piecewise-linear function $f_i(x)\equiv f(x)=\frac{1}{2}(|x+1|-|x-1|),\tau_1>0,\tau_2>0$. It is easy to prove the example 1 has unique equilibrium. By taking $c_1=c_2=9,a_{11}=b_{11}=2,a_{12}=-1,b_{12}=1,a_{22}=b_{22}=2,a_{21}=-1,b_{21}=1,I_1=14,I_2=5$ in theorem 3, then

$$c_1>|a_{11}|+|a_{21}|+|b_{11}|+|b_{21}|;c_2>|a_{12}|+|a_{22}|+|b_{12}|+|b_{22}|,$$

and so the unique equilibrium (2,1) is globally exponential stable.

Example 2. Consider the cellular neural networks with delays

$$\begin{cases} x_1'(t)=-7x_1(t)+f(x_1(t))-f(x_2(t))+2f(x_1(t-\tau_1))+f(x_2(t-\tau_2))+\sin t, \\ x_2'(t)=-8x_2(t)-f(x_1(t))+2f(x_2(t))+f(x_1(t-\tau_1))+f(x_2(t-\tau_2))+\cos t, \end{cases} \tag{7}$$

where the relation between the output of the cell and the state of the cell is described by a piecewise-linear function $f_i(x)\equiv f(x)=\frac{1}{2}(|x+1|-|x-1|),\tau_1>0,\tau_2>0$. By taking $c_1=7,c_2=8,a_{11}=1,b_{11}=2,a_{12}=-1,b_{12}=1,a_{22}=2,b_{22}=1,a_{21}=-1,b_{21}=1$ in theorem 6, we see that

$$c_1>|a_{11}|+|a_{21}|+|b_{11}|+|b_{21}|;c_2>|a_{12}|+|a_{22}|+|b_{12}|+|b_{22}|.$$

Thus by theorem 6, Eq. (7) has a unique 2π-periodic solution, and all other solutions of Eq. (7) converge exponentially to it as $t\to+\infty$.

V. CONCLUSION

In this paper, we have derived some simple sufficient conditions in term of systems parameters for global exponential stability and periodic solutions of delayed cellular neural networks (DCNNs), and the conditions possess highly important significance in some applied fields, for instance, they can be applied to design globally exponentially stable DCNNs and periodic oscillatory DCNNs and easily checked in practice by simple algebraic methods. These play an important role in the design and applications of DCNNs. In addition, the methods of this paper may be applied to some other systems such as the systems given in Refs. [10,15], and so on.

ACKNOWLEDGMENT

This work was supported by the Natural Science Foundation of Yunnan Province, China.

[1] L. O. Chua and L. Yang, IEEE Trans. Circuits Syst. **35**, 1257 (1988).
[2] L. O. Chua and L. Yang, IEEE Trans. Circuits Syst. **35**, 1273 (1988).
[3] T. Roska and L. O. Chua, Int. J. Circuit Theory Appl., **20**, 469 (1992).
[4] T. Roska, C.W. Wu, and L. O. Chua, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **40**, 270 (1993).
[5] T. Roska, C.W. Wu, M. Balsi, and L. O. Chua, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **39**, 487 (1992).
[6] M. Gilli, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **41**, 518 (1994).
[7] Jinde Cao and Dongming Zhou, Neural Networks **11**, 1601 (1998).
[8] S. Arik and V. Tavsanoglu (unpublished).
[9] Jinde Cao, Phys. Rev. E **59**, 5940 (1999).
[10] K. Mastsuoka, Neural Networks **5**, 495 (1992).
[11] P. P. Civalleri and M. Gilli, IEEE Trans. Circuits Syst. **40**, 157 (1993).
[12] H. T. Lu and ZH. Y. He, Acta Electron. **25**, 1 (1997).
[13] X. X. Liao, Sci. China, Ser. A **24**, 1037 (1994).
[14] Jinde Cao, *Stability in Cellular Neural Networks with Delays*, *Proceedings of 14th IFAC'99 World Congress* (Elsevier, New York, 1999), Vol. E, pp. 71–74.
[15] Jinde Cao and Jibin Li, Appl. Math. Mech. **19**, 457 (1998).
[16] Jinde Cao (unpublished).